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1. (40 points) Check all the correct statements.
   - ◯ The stements $\neg(p \wedge (q \vee p))$ and $\neg p$ are equal.
   - ◯ The negation of the statement $(p \vee q) \wedge (q \vee \neg r)$ is equal to $(\neg p \wedge \neg q) \vee (\neg q \wedge r)$.
   - ◯ The sets $\{2k, -2k \mid k \in \mathbb{N}\}$ and $\{2k \mid k \in \mathbb{Z}\}$ are equal.
   - ◯ The sets $\{2k \mid k \in \mathbb{Z}\} \cup \{3k \mid k \in \mathbb{Z}\}$ and $\{6k \mid k \in \mathbb{Z}\}$ are equal.

**Solution:**

1. The statement is true since $p \wedge (q \vee p)$ is the same as $p$ (since if $p$ is true the statement is true and if $p$ is false the statement is false as well). Hence, $\neg(p \wedge (q \vee p))$ is the same as $\neg p$.
2. It is also true since
$$\neg((p \vee q) \wedge (q \vee \neg r)) = \neg(p \vee q) \vee \neg(q \vee \neg r) = (\neg p \wedge \neg q) \vee (\neg q \wedge r).$$
3. They are not the same since $0 = 2 \cdot 0 \in \{2k \mid k \in \mathbb{Z}\}$ but any $x \in \{2k, -2k \mid k \in \mathbb{N}\}$ has absolute value at least 2 i.e. $0 \notin \{2k, -2k \mid k \in \mathbb{N}\}$.
4. They are not the same since $2 \in \{2k \mid k \in \mathbb{Z}\} \cup \{3k \mid k \in \mathbb{Z}\}$ but $2 \notin \{6k \mid k \in \mathbb{Z}\}$.

2. (10 points) Let us consider three-points geometry, it is a theory with undefined terms: point, line, is on, and axioms:

   1. There exist exactly three points.
   2. Two distinct points are on exactly one line.
   3. Not all the three points are collinear i.e. they do not lay on the same line.
   4. Two distinct lines are on at least one point i.e. there is at least one point such that it is on both lines.

   Show that there are exactly three lines.

   **Solution:** Denote the points $p_1$, $p_2$, and $p_3$ (they exist by Axiom 1). By Axiom 2, there are lines $l_{1,2}$, $l_{1,3}$, and $l_{2,3}$ such that $p_i$ and $p_j$ are on $l_{i,j}$ $(i \neq j)$.

   Note that the lines $l_{1,2}$, $l_{1,3}$, and $l_{2,3}$ are different. Indeed, assume the opposite i.e. WLOG $l_{1,2} = l_{1,3}$. Note that $p_1$, $p_2$, and $p_3$ are on $l_{1,2}$ which contradicts Axiom 3.

   Let us now prove that there are no other lines. Assume the opposite i.e. that there is another line $l$. There is a point that is on $l$ and $l_{1,2}$. WLOG this point is $p_1$. Additionally there is a point $p_i$ $(i =\neq 1)$ that is on $l$ and $l_{2,3}$. However, it means that $p_1$ and $p_i$ are on $l$ which contradicts Axiom 2.

3. (10 points) Let $a_0 = 2$, $a_1 = 5$, and $a_n = 5a_{n-1} - 6a_{n-2}$ for all integers $n \geq 2$. Show that $a_n = 3^n + 2^n$ for all integers $n \geq 0$.

**Solution:** We prove this using induction by $n$. The base case for $n \leq 1$ is clear since $3^0 + 2^0 = 2$ and $3^1 + 2^1 = 5$.

Let us prove the induction step. Assume that $a_n = 3^n + 2^n$ and $a_{n-1} = 3^{n-1} + 2^{n-1}$, we need to prove that $a_{n+1} = 3^{n+1} + 2^{n+1}$. Note that

$$a_{n+1} = 5a_n - 6a_{n-1} = 5 \cdot 3^n + 5 \cdot 2^n - 6 \cdot 3^{n-1} - 6 \cdot 2^{n-1} = 3^{n-1} \cdot 9 + 2^{n-1}4 = 3^{n+1} + 2^{n+1}.$$

4. (10 points) Show that $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for all integers $n \geq 1$.

**Solution:** We prove the statement using induction by $n$. The base case for $n = 1$ is clear. Let us prove the induction step now. The induction hypothesis is $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$. Note that

$$1^2 + 2^2 + 3^2 + \cdots + n^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2 =$$
$$(n+1)\frac{2n^2 + n + 6n + 6}{6} = (n+1)\frac{(n+2)(2n+3)}{6}.$$

Hence, $1^2 + 2^2 + 3^2 + \cdots + n^2 + (n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6}$.