Name: ____________________

Pid: ____________________

Show all of your work. Full credit will be given only for answers with explanations.

1. (10 points) Find the maximum and minimum values of $f(x, y) = 4x^2 + 10y^2$ on the disk $x^2 + y^2 \leq 4$.

Solution: Let us find the critical points of $f(x, y)$, $\frac{\partial f}{\partial x} = 8x$ and $\frac{\partial f}{\partial y} = 20y$ so the only critical point is $\langle 0, 0 \rangle$. The value at this point os 0.

Let us now find the maximum and minimum on the border. Border of the disk is defined by the equation $x^2 + y^2 = 4$. Hence, we need to solve the following system of equations:

$$8x = \lambda 2x$$
$$20y = \lambda 2y$$
$$x^2 + y^2 = 4.$$

- If $x = 0$, then $y = 2$ and $\lambda = 10$.
- If $x \neq 0$, then $\lambda = 4$ and as a result, $y = 0$. Hence, $x = 2$.

As a result, the maximum is the maximum of 0, $4 \cdot 0^2 + 10 \cdot 2^2$, and $4 \cdot 2^2 + 10 \cdot 0^2$ which is 40. The minimum is the minimum of 0, $4 \cdot 0^2 + 10 \cdot 2^2$, and $4 \cdot 2^2 + 10 \cdot 0^2$ which is 0.

2. (10 points) Find $\iint_R x^2 + y^2 + xy dA$, where $R = [0,1] \times [1,2]$.

Solution: By Fubini's theorem the answer is equal to $\int_0^1 \int_1^2 x^2 + y^2 + xy\,dydx$. Note that $\int_1^2 x^2 + y^2 + xy\,dy = x^2y + \frac{1}{3}y^3 + \frac{1}{2}xy^2\Big|_{y=1}^{y=2} = x^2 + \frac{7}{3} + \frac{3}{2}x$. Hence, $\int_0^1 \int_1^2 x^2 + y^2 + xy\,dydx = \int_0^1 x^2 + \frac{7}{3} + \frac{3}{2}x\,dx = \frac{1}{3}x^3 + \frac{7}{3}x + \frac{3}{4}x^2\Big|_{x=0}^{x=1} = \frac{8}{3} + \frac{3}{4}$.

3. Consider the plane P with equation $z = 6x - 3y + 2$.

(a) (10 points) Find the equation of a plane parallel to P and passing through the point $\langle 1, 0, -1 \rangle$.

Solution: The equation should be of the form $6x - 3y - z = \ldots$ such that if we substitute $x = 1, y = 0, z = -1$ we get a true statement. Hence, the answer is $6x - 3y - z = 7$.

(b) (10 points) For which value of a is the vector $\langle -2, 1, a \rangle$ normal to the plane?

Solution: The vector $\langle 6, -3, -1 \rangle$ is a normal vector of P. Hence, the vector $\langle -2, 1, a \rangle$ is normal to P iff there is a real number λ such that $\lambda \langle 6, -3, -1 \rangle = \langle -2, 1, a \rangle$. This is possible only when $\lambda = -\frac{1}{3}$ and $a = \frac{1}{3}$.

4. Let $f(x, y) = \sin(x) + \sin(y)$.

(a) (5 points) Find the tangent planes at $\langle \pi, \pi, 0\rangle$ and $\langle \frac{\pi}{2}, \frac{\pi}{2}, 2\rangle$.

Solution: First we need to find the partial derivatives of f, $\frac{\partial f}{\partial x} = \cos(x)$ and $\frac{\partial f}{\partial y} = \cos(y)$.
Hence, $\frac{\partial f}{\partial x}(\pi, \pi) = \frac{\partial f}{\partial y}(\pi, \pi) = -1$ and $\frac{\partial f}{\partial x}(\frac{\pi}{2}, \frac{\pi}{2}) = \frac{\partial f}{\partial y}(\frac{\pi}{2}, \frac{\pi}{2}) = 0$.
As a result, the tangent planes are $z = -(x - \pi) - (y - \pi)$ and $z - 2 = 0$.

(b) (5 points) Check if these planes are intersecting; if they are intersecting, find symmetric equations for the line of intersection of the planes.

Solution: We need to find the intersection of the planes. Note that if we substitute $z = 2$ to the first equation we get $2 - 2\pi = -x - y$. As a result a line defined by the equations

$$\begin{aligned} 2 - 2\pi + x = y \\ z = 2 \end{aligned}$$

is the intersection of the planes.

5. Let $f(x, y) = 2xy$ and $g(x, y)$ be the maximum value of $D_u f(x, y)$ over all unit vectors u.

   (a) (10 points) Find the value of $g(1, 3)$.

   Solution: We proved in class that the maximum value of $D_u f(x, y)$ is equal to $|\nabla f(x, y)|$. In other words, $g(x, y) = |\nabla f(x, y)|$. Note that $\nabla f(x, y) = \langle 2y, 2x \rangle$. As a result, $g(x, y) = 2\sqrt{x^2 + y^2}$ and $g(1, 3) = 2\sqrt{10}$.

   (b) (10 points) Find and classify all the critical points of $g(x, y)$.

   Solution: In order to find the critical points of g we need to find the partial derivatives, $\frac{\partial g}{\partial x} = \frac{2x}{\sqrt{x^2+y^2}}$ and $\frac{\partial g}{\partial y} = \frac{2y}{\sqrt{x^2+y^2}}$. We may note that $\frac{\partial g}{\partial x}(x, y) = 0$ iff $x = 0$ and $\frac{\partial g}{\partial y} = 0$ iff $y = 0$, but the derivatives are not defined at $\langle 0, 0 \rangle$. As a result, there are no critical points.

6. Let $r = \langle u+v, u+v^2, u^2+v \rangle$, where $u = \cos(x) + \cos(\pi \cdot y)$ and $v = \sin(xy)$.

(a) (5 points) Find $\frac{\partial r}{\partial x}$ and $\frac{\partial r}{\partial y}$.

Solution: Let us use the chain rule, $\frac{\partial r}{\partial x} = \frac{\partial r}{\partial u}\frac{\partial u}{x} + \frac{\partial r}{\partial v}\frac{\partial v}{x} = -\sin(x)\langle 1, 1, 2u\rangle + y\cos(xy)\langle 1, 2v, 1\rangle$ and $\frac{\partial r}{\partial y} = \frac{\partial r}{\partial u}\frac{\partial u}{y} + \frac{\partial r}{\partial v}\frac{\partial v}{y} = -\pi\sin(\pi y)\langle 1, 1, 2u\rangle + x\cos(xy)\langle 1, 2v, 1\rangle$.

(b) (5 points) Find the tangent plane of the surface described by the vector function r for $x = \frac{\pi}{3}$ and $y = 1$.

Solution: If $x = \frac{\pi}{3}$ and $y = 1$, then $u = \frac{1}{2} - 1 = -\frac{1}{2}$ and $v = \frac{\sqrt{3}}{2}$. Hence, $\frac{\partial r}{\partial x} = -\frac{\sqrt{3}}{2}\langle 1, 1, -1\rangle + \frac{1}{2}\langle 1, \sqrt{3}, 1\rangle = \frac{1}{2}\langle 1-\sqrt{3}, 0, 1+\sqrt{3}\rangle$ and $\frac{\partial r}{\partial y} = \frac{\pi}{6}\langle 1, \sqrt{3}, 1\rangle$.

In order to find a normal vector to the plane we need to compute $\langle 1, \sqrt{3}, 1\rangle \times \langle 1-\sqrt{3}, 0, 1+\sqrt{3}\rangle = \langle 3+\sqrt{3}, -2\sqrt{3}, 3-\sqrt{3}\rangle$. Additionally, $r = \langle \frac{\sqrt{3}-1}{2}, \frac{3}{4} - \frac{1}{2}, \frac{\sqrt{3}}{2} + \frac{1}{4}\rangle$.

As a result, the answer is $0 = (3+\sqrt{3})(x - \frac{\sqrt{3}-1}{2}) - 2\sqrt{3}(y - \frac{3}{4} + \frac{1}{2}) + (3-\sqrt{3})(z - \frac{\sqrt{3}}{2} - \frac{1}{4})$.

7. (10 points) Find the linear approximation of the function $f(x, y) = x^2 + yx$ at $\langle 1, -1\rangle$.

Solution: Let us compute the partial derivatives, $\frac{\partial f}{\partial x} = 2x + y$ and $\frac{\partial f}{\partial y} = x$. Hence, $\frac{\partial f}{\partial x}(1, -1) = 1$ and $\frac{\partial f}{\partial y}(1, -1) = 1$.

As a result the linear approximation of f at $\langle 1, -1\rangle$ is equal to $(x - 1) + (y + 1)$ since $f(1, -1) = 0$.