Name: ______________________

Pid: ______________________

1. (10 points) In the subtraction game with *two* piles where players may subtract 1, 2 or 5 chips on their turn, identify the N- and P-positions. (Please do not forget to prove correctness of your asnwer.)

Solution: Let us prove that

$$g(x) = \begin{cases} 0 & \text{if } x \equiv 0 \pmod 3 \\ 1 & \text{if } x \equiv 1 \pmod 3 \\ 2 & \text{if } x \equiv 2 \pmod 3 \end{cases}$$

is the Sprague–Grundy function for the subtraction game with *one* pile.

We prove the statement using induction by x. First we need to prove the base cases for $x \leq 4$.

- It is clear that $g(0) = 0$ since 0 is a terminal position.
- From 1 there is only one move to 0; hence, $g(1) = \text{mex}\{0\} = 1$.
- From 2 there are two moves to 0 and to 1; hence, $g(2) = \text{mex}\{0, 1\} = 2$.
- From 3 there are two moves to 1 and to 2; hence, $g(3) = \text{mex}\{1, 2\} = 0$.
- From 4 there are two moves to 0 and to 1; hence, $g(4) = \text{mex}\{2, 0\} = 1$.

Assume that the statement is true for all $y < x$. Note that there are three moves from x: to $x-1$, to $x-2$, and to $x-5$. It is easy to see that $x-5 \equiv x-2 \pmod 3$; hence $g(x) = \text{mex}\{g(x-1), g(x-2)\}$. Therefore, by considering three cases of the reminder of x modulo 3 we can prove the statement.

2. (10 points) Alice and Bob have several piles of chips. On each turn they can either remove 1 or 2 chips from one pile, or split a pile into two *nonempty* piles. Players take turns and a player that cannot make a move loses. Find the value of the Sprague–Grundy function for positions with one pile made of n chips. (Please do not forget to prove correctness of your asnwer.)

Solution: Let g be the Sprague–Grundy function for this game. It is clear that the position (x, y) (the position with two piles having x and y chips, respectively) is equivalent to the position (x, y) in the same of this game with itself. Hence, $g(x, y) = g(x) \oplus g(y)$.

As a result,

$$g(x)\text{mex}(\{g(x-1), g(x-2)\} \cup \{g(y) \oplus g(z) \ : \ y, z \geq 1, y + z = x\})$$

for $x \geq 2$ and $g(0) = 1$ and $g(1) = 1$.

Let us prove that

$$g(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \equiv 1 \pmod 4 \\ 2 & \text{if } x \equiv 2 \pmod 4 \\ 0 & \text{if } x \equiv 3 \pmod 4 \\ 3 & \text{if } x \equiv 0 \pmod 4 \end{cases}.$$

The base case for $x \leq 4$ is clear.

- By the above formula, $g(2) = \text{mex}\{g(0), g(1), g(1) \oplus g(1)\} = \text{mex}\{0, 1\} = 2$.
- By the above formula, $g(3) = \text{mex}\{g(1), g(2), g(1) \oplus g(2)\} = \text{mex}\{1, 2, 3\} = 0$.
- By the above formula, $g(4) = \text{mex}\{g(2), g(3), g(1) \oplus g(3), g(2) \oplus g(2)\} = \text{mex}\{2, 0, 1, 0\} = 3$.

Let us now prove the induction step. Assume the statement is true for all $y < x$.

- Let $x \equiv 1 \pmod 4$ Assume $y + z = x$ and $y, z \geq 1$. We calim that $g(y) \oplus g(z) \neq 1$. Indeed, the only pairs of numbers whose xor gives 1 among $0, 1, 2, 3$ are 0 and 1, and 2 and 3.
  - If $g(y) = 0$ and $g(z) = 1$, then $y \equiv 3 \pmod 4$ and $z \equiv 1 \pmod 4$. Which implies that $y + z \equiv 0 \pmod 4$ and this contradicts to the assumption.
  - If $g(y) = 2$ and $g(z) = 3$, then $y \equiv 2 \pmod 4$ and $z \equiv 0 \pmod 4$. Which implies that $y + z \equiv 2 \pmod 4$ and this contradicts to the assumption.

  However, $g(x - 1) = 3$ and $g(x - 2) = 0$. Hence, $g(x) = 1$.
- Let $x \equiv 2 \pmod 4$ Assume $y + z = x$ and $y, z \geq 1$. We calim that $g(y) \oplus g(z) \neq 2$. Indeed, the only pairs of numbers whose xor gives 2 among $0, 1, 2, 3$ are 0 and 2, and 1 and 3.
  - If $g(y) = 0$ and $g(z) = 2$, then $y \equiv 3 \pmod 4$ and $z \equiv 2 \pmod 4$. Which implies that $y + z \equiv 1 \pmod 4$ and this contradicts to the assumption.
  - If $g(y) = 1$ and $g(z) = 3$, then $y \equiv 1 \pmod 4$ and $z \equiv 0 \pmod 4$. Which implies that $y + z \equiv 1 \pmod 4$ and this contradicts to the assumption.

  However, $g(x - 1) = 1$, $g(x - 2) = 3$, and $g(x - 1) \oplus g(1) = 0$. Hence, $g(x) = 2$.
- Let $x \equiv 3 \pmod 4$ Assume $y + z = x$ and $y, z \geq 1$. We calim that $g(y) \oplus g(z) \neq 0$. Indeed, the only pairs of numbers whose xor gives 3 among $0, 1, 2, 3$ are the equal pairs

- If $g(y) = 0$ and $g(z) = 0$, then $y \equiv 3 \pmod 4$ and $z \equiv 3 \pmod 4$. Which implies that $y + z \equiv 2 \pmod 4$ and this contradicts to the assumption.
- If $g(y) = 1$ and $g(z) = 1$, then $y \equiv 1 \pmod 4$ and $z \equiv 1 \pmod 4$. Which implies that $y + z \equiv 2 \pmod 4$ and this contradicts to the assumption.
- If $g(y) = 2$ and $g(z) = 2$, then $y \equiv 2 \pmod 4$ and $z \equiv 2 \pmod 4$. Which implies that $y + z \equiv 0 \pmod 4$ and this contradicts to the assumption.
- If $g(y) = 3$ and $g(z) = 3$, then $y \equiv 0 \pmod 4$ and $z \equiv 0 \pmod 4$. Which implies that $y + z \equiv 0 \pmod 4$ and this contradicts to the assumption.

However, $g(x - 1) = 2$, $g(x - 2) = 1$.

- Let $x \equiv 0 \pmod 4$ Assume $y + z = x$ and $y, z \geq 1$. We calim that $g(y) \oplus g(z) \neq 3$. Indeed, xor of two numbers among $0, 1, 2, 3$ is at most 3. However, $g(x - 1) = 0$, $g(x - 2) = 2$, and $g(x - 1) \oplus g(1) = 1$. Hence, $g(x) = 3$.