Chapter 10

The Pigeonhole Principle

The principle we are going to discuss in this chapter is very simple: it states that if you have more objects than boxes, then you cannot put all the objects to boxes without puting two objects in the same box.

More formally the principle can be formulated as follows: if $n > m$, then any function from $[n]$ to $[m]$ is not an injection. This simple statement is famous in mathematics and called *the pigeonhole principle*[1].

Theorem 10.1. *Let X and Y be some sets such that $|X| > |Y|$. Then for any function $f : |X| \to |Y|$ there are $x_0 \neq x_1 \in X$ such that $f(x_0) = f(x_1)$.*

Proof. The statement follows from Theorem 8.6. □

This simple statement is very handy in combinatorics. For example, using this statement one may prove that in any group of more than 12 people there are two people who were born in the same month.

Assume that there are n people in the group and $n > 12$. Consider the following function $f : [n] \to [12]$ such that $f(i) = j$ if the ith person was born in jth month. Note that f is not an injection since $n > 12$ i.e. there are $i_0 \neq i_1$ such that i_0th and i_1th person are born in the same month.

We may also prove that in any group of people there are two people who are friends with the same number of people in the group.

Assume the number of people is n. It is easy to see that every person may have at most $n - 1$ friends. Hence, we may define a function $f : [n] \to \{0, \ldots, n-1\}$ such that $f(i)$ is equal to the number of friends in this group of the ith person in this group. We need to consider two cases.

- If $\mathrm{Im} f \subseteq [n-1]$. In this case $|[n]| > |\mathrm{Im} f|$ and f is not an injection.
- Otherwise, note that it is not possible that $(n-1) \in \mathrm{Im} f$ since it there is a friend of nobody it is not possible that there is a friend of everyone. Hence, $f : [n] \to \{0, 1, \ldots, n-2\}$ and f is not an injection.

[1] The pigeonhole principle is also called the Dirichlet principle, after the German mathematician G. Lejeune Dirichlet, who demonstrated, using this principle, that there were at least two Parisians with the same number of hairs on their heads.

Theorem 10.2 (Erdős—Szekeres)**.** *Every sequence of $(r-1)(s-1)+1$ distinct real numbers contains a subsequence of length r that is increasing or a subsequence of length s that is decreasing.*

Proof. Given a sequence of length $(r-1)(s-1)+1$, label each number x_i in the sequence with the pair (a_i, b_i), where a_i is the length of the longest increasing subsequence ending with x_i and b_i is the length of the longest decreasing subsequence ending with x_i. Each two numbers in the sequence are labeled with a different pair: if $i < j$ and $x_i < x_j$ then $a_i < a_j$, and on the other hand if $x_i > x_j$ then $b_i < b_j$. But there are only $(r-1)(s-1)$ possible labels if a_i is at most $r-1$ and b_i is at most $s-1$, so by the pigeonhole principle there must exist a value of i for which a_i or bi is outside this range. If ai is out of range then x_i is part of an increasing sequence of length at least r, and if b_i is out of range then x_i is part of a decreasing sequence of length at least s. □

10.1 The Generalized Pigeonhole Principle

One may generalize the pigeonhole principle in the following way. If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

Theorem 10.3. *Let X and Y be some sets. Then for any function $f : |X| \to |Y|$ there are $x_1, \ldots, x_\ell \in X$ such that*

- $f(x_i) = f(x_j)$,
- $x_i \neq x_j$ *for any* $i \neq j \in [\ell]$, *and*
- $\ell \geq \lceil |X|/|Y| \rceil$

Exercise 10.1. *Prove Theorem 10.3.*

Using this theorem we can prove that if we draw 9 cards out of a deck of cards, we are guaranteed that at least three of them are of the same suit. Indeed, there are 4 suits and by pigeonhole principle if we put each card to one out of four boxes according to their suit, one of the boxes should have at least $\lceil 9/4 \rceil = 3$ cards.

Another example shows how the generalized pigeonhole principle can be applied to an important part of combinatorics called Ramsey theory.

Assume that in a group of six people, each pair of individuals consists of two friends or two enemies. One may prove that there are either three mutual friends or three mutual enemies in the group.

Let A be one of the six people; of the five other people in the group, there are either three or more who are friends of A, or three or more who are his enemies A. This statements follows from the generalized pigeonhole principle since when five objects are divided into two sets, one of the sets has at least $\lceil 5/2 \rceil = 3$ elements. Without loss of generality we may suppose that B, C, and D are friends of A. If any two of these three individuals are friends, then these two and A form a group of three mutual friends. Otherwise, B, C, and D form a set of three mutual enemies.

End of The Chapter Exercises

10.2 Show that among any group of five (not necessarily consecutive) integers, there are two with the same remainder when divided by 4.

10.3 Show that if there are 30 students in a class, then at least two have last names that begin with the same letter.

10.4 Let n be a positive integer. Show that in any set of n consecutive integers there is exactly one divisible by n.

10.5 Prove that for every integers $a_1, \ldots, a_n$ there are $k > 0$ and $\ell \geq 0$ such that $k + \ell \leq n$ and $\sum_{i=k}^{k+\ell} a_i$ is divisible by n.

10.6 Let $S \subseteq [20]$ be a set. Show that if $|S| \geq 13$, then there are $a, b \in S$ such that $a - b = 6$.

10.7 How many numbers must be selected from the set [6] to guarantee that at least one pair of these numbers add up to 7?

10.8 Sasha is training for a triathlon. Over a 30 day period, he pledges to train at least once per day, and 45 times in all. Then there will be a period of consecutive days where he trains exactly 14 times.

10.9 Show that among any $n+1$ positive integers not exceeding $2n$ there must be an integer that divides one of the other integers.

10.10 Let $a_1, a_2, \ldots, a_t$ be positive integers. Show that if $a_1+a_2+\cdots+a_t-t+1$ objects are placed into t boxes, then for some $i \in [t]$, the ith box contains at least a_i objects.

10.11 Let $\{(x_1, y_1), \ldots, (x_5, y_5)\} \subseteq \mathbb{Z}^2$ be a set of five distinct points with integer coordinates in the xy plane. Show that the midpoint of the line joining at least one pair of these points has integer coordinates.