Chapter 4

Predicates and Connectives

4.1 Propositions and Predicates

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Connectives and Propositions

In the previous chapters we used the word "statement" without any even relatively formal definition of what it means. In this chapter we are going to give a semi-formal definition and discuss how to create complicated statements from simple statements.

It is difficult to give a formal definition of what a mathematical statement is, hence, we are not going to do it in this book. The goal of this section is to enable the reader to recognize mathematical statements.

A *proposition* or a mathematical statement is a declarative sentence which is either true or false but not both. Consider the following list of sentences.

1. $2 \times 2 = 4$
2. $\pi = 4$
3. n is even
4. 32 is special
5. The square of any odd number is odd.
6. The sum of any even number and one is prime.

Of those, the first two are propositions; note that this says nothing about whether they are true or not. Actually, the first is true and the second is false. However, the third sentence becomes a proposition only when the value of n is fixed. The fourth is not a proposition. Finally, the last two are propositions (the fifth is true and the sixth is false).

The third statement is somewhat special, because there is a simple way to make it a proposition: one just needs to fix the value of the variables. Such sentences are called predicates and the variables that need to be specified are called free variables of these predicates.

Note that the fourth sentence is also interesting, since if we define what it means to be special, the phrase became a proposition. Mathematicians tend to do such things to give mathematical meanings to everyday words.

4.2 Connectives

Mathematicians often need to decide whether a given proposition is true or false. Many statements are complicated and constructed from simpler statements using *logical connectives*. For example we may consider the following statements:

1. $3 > 4$ and $1 < 1$;
2. $1 \times 2 = 5$ or $6 > 1$.

Logical connective "OR". The second statement is an example of usage of this connective. The statement "P or Q" is true if and only if at least one of P and Q is true. We may define the connective using the truth table of it.

P	Q	P or Q
T	T	T
T	F	T
F	T	T
F	F	F

The or connective is also called *disjunction* and the disjunction of P and Q is often dented as $P \vee Q$.

Warning: Note that in everyday speech "or" is often used in the exclusive case, like in the sentence "we need to decide whether it is an insect or a spider". In this case the precise meaning of "or" is made clear by the context. However, mathematical language should be formal, hence, we always use "or" inclusively.

Logical connective "AND". The first statement is an example of this connective. The statement "P and Q" is true if and only if both P and Q are true. We may define the connective using the truth table of it.

P	Q	P and Q
T	T	T
T	F	F
F	T	F
F	F	F

The or connective is also called *conjunction* and the conjunction of P and Q is often dented as $P \wedge Q$.

Warning: Not all the properties of "and" from everyday speach are captured by logical conjunction. For example, "and" sometimes implies order. For example, "They got married and had a child" in common language means that the marriage came before the child. The word "and" can also imply a partition of a thing into parts, as "The American flag is red, white, and blue." Here it is not meant that the flag is at once red, white, and blue, but rather that it has a part of each color.

Logical connective "NOT". The last connective is called *negation* and examples of usage of it are the following:

1. 5 is not greater than 8;
2. Does not exist an integer n such that $n^2 = 2$.

Note that it is not straightforward where to put the negation in these sentences.

The negation of a statement P is denoted as $\neg P$ (sometimes it is also denoted as $\sim P$).

End of The Chapter Exercises

4.1 Construct truth tables for the statements

- not (P and Q);
- (not P) or (not Q);
- P and (not Q);
- (not P) or Q;

4.2 Consider the statement "All gnomes like cookies". Which of the following statements is the negation of the above statement?

- All gnomes hate cookies.
- All gnomes do not like cookies.
- Some gnome do not like cookies.
- Some gnome hate cookies.
- All creatures who like cookies are gnomes.
- All creatures who do not like cookies are not gnomes.

4.3 Using truth tables show that the following statements are equivalent:

- $P \implies Q$,

- $(P \vee Q) \iff Q$ ($A \iff B$ is the same as $(A \implies B) \wedge (B \implies A)$),
- $(P \wedge Q) \iff P$

4.4 Prove that three connectives "or", "and", and "not" can all be written in terms of the single connective "notand" where "P notand Q" is interpreted as "not (P and Q)" (this operation is also known as Sheffer stroke or NAND).

4.5 Show the same statement about the connective "notor" where "P notor Q" is interpreted as "not (P or Q)" (this operation is also known as Peirce's arrow or NOR).