Chapter 7

Relations

Nonetheless that function are used almost everywhere in mathematics, many relations are not functional by their nature. For example, could never define a function $r(a)$ that gives the solution of $x^2 = a$ because there are two solutions for $a > 0$ and there are zero solutions for $a < 0$. A relation is a more general mathematical object.

In order to define a relation we need to relax the definition of the graph of a function (Section 6.3) by allowing more than one "result" and by allowing zero "results". In other words we just say that any set $R \subseteq X_1 \times \cdots \times X_k$ is a k-ary *relation* on $X_1, \ldots, X_k$. We also say that $x_1 \in X_1, \ldots, x_k \in X_k$ are in the relation R iff $(x_1, \ldots, x_k) \in R$. If $k = 2$ such a relation is called a *binary relation* and we write xRy if x and y are in the relation R. If $X_1 = \cdots = X_k = X$, we say that R is a k-ary relation on X.

Note that $=$, $\leq$, $\geq$, $<$, and $>$ define relations on $\mathbb{R}$ (or any subset S of $\mathbb{R}$). For example, if $S = \{0, 1, 2\}$, then $<$ defines the relation $R = \{(0,1), (0,2), (1,2)\}$.

Probably the most popular relation in mathematics is the following relation on $\mathbb{Z}$. Let $a, b \in \mathbb{Z}$. If n divides $a - b$ for some $n \in Z$, we say that "a equivalent to b modulo n" and denote it as $a \equiv b \pmod{n}$. For example, 1 and 4 are equivalent modulo 3 since 3 divides $1 - 4 = -3$.

7.1 Equivalence Relations

The definition of a relation is way to broad. Hence, quite often we consider some types of relation. Probably the most interesting type of the relations is equivalence relations.

Definition 7.1. *Let R be a relation on a set X. We say that R is an equivalence relation if it satisfies the following conditions:*

reflexivity: *xRx for any $x \in X$;*

symmetry: *xRy iff yRx for any $x, y \in X$;*

transitivity: *for any $x, y, z \in X$, if xRy and yRZ, then xRz;*

One may guess that the equivalence relation are mimicking $=$, so it is not a surprise that $=$ is an equivalence relation.

The definition seems quite bizarre, however, all of you are already familiar with an important example: you know that equivalent fractions represent the same number. For example $\frac{2}{4}$ is the same as $\frac{1}{2}$. Let us consider this example more thorough, let S be a set of symbols of the form $\frac{x}{y}$ (note that it is not a set of numbers) where $x, y \in Z$ and $y \neq 0$. We define a binary relation R on S such that $\frac{x}{y}$ and $\frac{z}{w}$ are in the relation R iff $xw = zy$. It is easy to prove that this relation is an equivalence relation.

reflexivity: Let $\frac{a}{b} \in S$. Since $ab = ab$, we have that $\frac{a}{b} R \frac{a}{b}$.

symmetry: Let $\frac{a}{b}, \frac{c}{d} \in S$. Suppose that $\frac{a}{b} R \frac{c}{d}$, by the definition of R, it implies that $ac = db$. As a result, $\frac{c}{d} R \frac{a}{b}$.

transitivity: Let $\frac{a}{b}, \frac{c}{d}, \frac{e}{f} \in S$ with $\frac{a}{b} R \frac{c}{d}$ and $\frac{c}{d} R \frac{e}{f}$. Then $ad = cb$ and $cf = ed$. The first equality can be rewritten as $c = ad/b$. Hence, $adf/b = ed$ and $af = eb$ since $d \neq 0$. So $\frac{a}{b} R \frac{e}{f}$.

7.1.1 Partitions

Let S be some set. We say that $\{P_1, \ldots, P_k\}$ form a partition of S iff $P_1, \ldots, P_k$ are pairwise disjoint and $P_1 \cup \cdots \cup P_k = S$; in other words, a partition is a way of dividing a set into overlapping pieces.

Exercise 7.1. *Let $\{P_1, \ldots, P_k\}$ be a partition of a set S and R be a binary relation of S such that aRb iff $a, b \in P_i$ for some $i \in [k]$. Show that R is an equivalence relation.*

This exercise shows that one may transform a partition of the set S into an equivalence relation on S. However, it is possible to do the opposite.

Theorem 7.1. *Let R be a binary equivalence relation on a set S. For any element $x \in S$, define $R_x = \{y \in S : xRy\}$ (the set of all the elements of S related to x) we call such a set the equivalence class of x. Then $\{R_x : x \in S\}$ is a partition of S.*

Exercise 7.2. *Prove Theorem 7.1.*

7.1.2 Modular Arithmetic

The relation "$\equiv \pmod n$" is actively used in the number theory. One of the important properties of this relation is that it is an equivalence relation.

Theorem 7.2. *The relation $\equiv \pmod n$ is an equivalence relation.*

Proof. To prove this statement we need to prove all three properties: reflexivity, symmetry, and transitivity.

reflexivity: Note that for any integer x, $x - x = 0$ is divisible by any integer including n. Hence, $x \equiv x \pmod{n}$.

symmetry: Let us assume that $x \equiv y \pmod{n}$; i.e. $x - y = kn$ for some integer k. Note that $y - x = (-k)n$, so $y \equiv x \pmod{n}$.

transitivity: finally, assume that $x \equiv y \pmod{n}$ and $y \equiv z \pmod{n}$; i.e. $x - y = kn$ and $y - z = \ell n$ for some integers k and ℓ. It is easy to note that $x - z = (x - y) + (y - z) = (k + \ell)n$. As a result, $x \equiv z \pmod{n}$.

Thus, we proved that $\equiv \pmod{n}$ is an equivalence relation. □

Let $x \in \mathbb{Z}$; we denote by $r_{x,n}$ the equivalence class of x with respect to the relation $\equiv \pmod{n}$, we also denote by $\mathbb{Z}/n\mathbb{Z}$ the set of all the equivalence classes with respect to the relation $\equiv \pmod{n}$.

Another important property of these relation is that they behave well with respect to the arithmetic operations.

Theorem 7.3. *Let $x, y \in Z$ and $n \in \mathbb{N}$. Suppose that $a \in r_{x,n}$ and $b \in r_{y,n}$, then $(a + b) \in r_{x+y,n}$ and $ab \in r_{xy,n}$.*

Using this theorem we may define arithmetic operations on the equivalence classes with respect to the relation $\equiv \pmod{n}$. Let $x, y \in \mathbb{Z}$ and $n \in \mathbb{N}$. Then $r_{x,n} + r_{y,n} = \{a + b \ : \ a \in r_{x,n}, b \in r_{y,n}\} = r_{x+y,n}$ and $r_{x,n}r_{y,n} = \{ab \ : \ a \in r_{x,n}, b \in r_{y,n}\} = r_{xy,n}$. Moreover, these operations have plenty of good properties.

Exercise 7.3. *Let $a, b, c \in \mathbb{Z}/n\mathbb{Z}$. Show that the following equalities are true:*

- $a + (b + c) = (a + b) + c$,
- $a + r_{0,n} = a$ *(thus we denote $r_{0,n}$ as 0)*,
- $ar_{1,n} = a$ *(thus we denote $r_{1,n}$ as 1)*,
- *there is a class $d \in \mathbb{Z}/n\mathbb{Z}$ such that $a + d = r_{0,n}$ (thus we denote this d as $-a$)*,
- $a + b = b + a$,
- $ab = ba$,
- $a(b + c) = ab + ac$,

7.2 Partial Orderings

In the previous section we discussed a mathematical way to express the property being similar. In this section we are going to give a way to analyze relation similar to comparisons.

Definition 7.2. *A binary relation R on S is a partial ordering if it satisfies the following constraints.*

reflexivity: *xRx for any $x \in S$;*

antisymmetry: *if xRy and yRx, then $x = y$ for all $x, y \in S$;*

transitivity: *for any $x, y, z \in S$, if xRy and yRZ, then xRz;*

We say that an order R on a set S is total iff for any $x, y \in S$, either xRy or yRx.

Note that if S is a set of numbers, then $\leq$ defines a partial ordering on S; moreover, it defines a total order.

Typically we use symbols similar to $\preceq$ to denote partial orderings and we write $a \prec b$ to express that $a \preceq b$ and $a \neq b$.

Let $|$ be the relation on $\mathbb{Z}$ such that $d \mid n$ iff d divides n.

Theorem 7.4. *The relation $|$ is a partial ordering of the set $\mathbb{N}$.*

Proof. To prove that this relation is a partial ordering we need to check all three properties.

reflexivity: Note that $x = 1 \cdot x$ for any integer x; hence, $x \mid x$ for any integer x.

antisymmetry: Assume that $x \mid y$ and $y \mid x$. Note that it means that $kx = y$ and $\ell y = x$ for some integers k and ℓ. Hence, $y = (k \cdot \ell)y$ which implies that $k \cdot \ell = 1$ and $k = \ell = 1$. Thus, $x = y$.

transitivity: finally, assume that $x \mid y$ and $y \mid z$; i.e. $kx = y$ and $\ell y = z$. As a result, $(k \cdot \ell)x = z$ and $x \mid z$.

□

Exercise 7.4. *Let S be some set, show that $\subseteq$ defines a partial ordering on the set 2^S.*

7.2.1 Topological Sorting

Partial orderings are very useful for describing complex processes. Suppose that some process consists of several tasks, T denotes the set of these tasks. Some tasks can be done only after some others e.g. when you cooking a salat you need to wash vegetables before you chop them. If $x, y \in T$ be some tasks, $x \preceq y$ if x should be done before y and this is a partial ordering.

In the applications this order is not a total order because some steps do not depend on other steps beeing done first (you can chop tomatos and chop cucumbers in any order). However, if we need to create a schedule in which the tasks should be done, we need to create a total ordering on T. Moreover, this order should be compatible with the partial ordering. In other words, if $x \preceq y$, then $x \preceq_t y$ for all $x, y \in T$, where $\preceq_t$ is the total order. The technique of finding such a total ordering is called *topological sorting.*

Theorem 7.5. *Let S be a finite set and $\preceq$ be a partial order on S. Then there is a total order $\preceq_t$ on S such that if $x \preceq y$, then $x \preceq_t y$ for all $x, y \in S$*

This sorting can be done using the following procedure.

- Initiate the set S beeing equal to T
- Choose the minimal element of the set S with respect to the ordering $\preceq$ (such an element exists since S is a finite set, see Chapter 8). Add this element to the list, remove it from the set S, and repeate this step if $S \neq \emptyset$.

Let us consider the following example. In the left column we list the classes and in the right column the prerequisite.

Courses	Prerequisite
Math 20A	
Math 20B	Math 20A
Math 20C	Math 20B
Math 18	
Math 109	Math 20C, Math 18
Math 184A	Math 109

We need to find an order to take the courses.

1. We start with
$$S = \{\text{Math 20A}, \text{Math 20B}, \text{Math 20C}, \text{Math 18}, \text{Math 109}, \text{Math 184}\}.$$
There are two minimal elements: Math 20A and Math 18. Let us remove Math 18 from S and add it to the resulting list R.

2. Now we have
$$R = \text{Math 18}$$
and
$$S = \{\text{Math 20A}, \text{Math 20B}, \text{Math 20C}, \text{Math 109}, \text{Math 184}\}.$$
There is only one minimal element Math 20A. We remove it and add it to the list R.

3. On this step
$$R = \text{Math 18}, \text{Math 20A}$$
and
$$S = \{\text{Math 20B}, \text{Math 20C}, \text{Math 109}, \text{Math 184}\}.$$
Again there is only one minimal element: Math 20B.

4. $$R = \text{Math 18}, \text{Math 20A}, \text{Math 20B}$$
and
$$S = \{\text{Math 20C}, \text{Math 109}, \text{Math 184}\}.$$
There is only one minimal element: Math 20C.

5. $$R = \text{Math 18}, \text{Math 20A}, \text{Math 20B}, \text{Math 20C}$$

   and
   $$S = \{\text{Math 109}, \text{Math 184}\}.$$

   There is only one minimal element: Math 109.

6. Finally,
   $$R = \text{Math 18}, \text{Math 20A}, \text{Math 20B}, \text{Math 20C}, \text{Math 109}$$

   and
   $$S = \{\text{Math 184}\}.$$

   There is only one minimal element: Math 184A.

As a result, the final list is

$$R = \text{Math 18}, \text{Math 20A}, \text{Math 20B}, \text{Math 20C}, \text{Math 109}, \text{Math 184A}.$$

End of The Chapter Exercises

7.5 Show that the relation $|$ does not define a partial ordering on $\mathbb{Z}$.

7.6 Let a relation R be defined on the set of real numbers as follows: xRy iff $2x + y = 3$. Show that it is antisymmetric.

7.7 Are there any minimal elements in $\mathbb{N}$ with respect to $|$? Are there any maximal elements?