

Chapter 9

Counting Principles

9.1 The Additive Principle



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Counting Principles

The first principle is called *additive* principle and it states that if you have two disjoint sets, then their union have size equal to the sum of their sizes.

A simple illustration of this statement is the following. Assume you have three pencils and two pens; how many ways to choose a writing accessory. According to this principle the answer is $2 + 3 = 5$.

Theorem 9.1 (The Additive Principle). *Let X and Y be finite sets. If $X \cap Y = \emptyset$, then $|X \cup Y| = |X| + |Y|$.*

Proof. Let $|X| = n$, $|Y| = m$ and $g : [n] \rightarrow X$ and $h : [m] \rightarrow Y$ be bijections. In order to prove it we just construct a bijection $f : [n + m] \rightarrow (X \cup Y)$.

$$f(i) = \begin{cases} g(i) & i < n \\ h(i - n) & i > n \end{cases}.$$

It's easy to see that f is an injection. Indeed, assume the opposite i.e. that there are $i_0 \neq i_1 \in X \cup Y$ such that $f(i_0) = f(i_1)$. There are three cases.

- The first is when $i_0, i_1 \in [n]$. In this case $g(i_0) = g(i_1)$ which contradicts the assumption that g is a bijection.
- The second is when $i_0, i_1 \in \{n + 1, n + 2, \dots, m\}$. In this case $h(i_0 - n) = h(i_1 - n)$ which contradicts the assumption that h is a bijection.
- Finally, the last case is when $i_0 \in [n]$ and $i_1 \in \{n + 1, n + 2, \dots, m\}$. It is easy to see that this implies that $g(i_0) = h(i_1 - n)$. However, it means that $g(i_0) = h(i_1 - n) \in (X \cap Y)$, which contradicts the assumption that $X \cap Y = \emptyset$.

To finish the proof we need to show that f is a surjection. Let $w \in (X \cup Y)$. Consider the following two cases.

- Let $w \in X$. There is $i \in [n]$ such that $f(i) = g(i) = w$ since g is a bijection.
- Otherwise, $w \in Y$. In this case, there is $i \in [m]$ such that $f(i+n) = h(i) = w$ since h is a bijection.

□

Corollary 9.1. *Let X_1, \dots, X_n be some pairwise disjoint sets. Then $|\bigcup_{i=1}^n X_i| = \sum_{i=1}^n |X_i|$.*

Exercise 9.1. *Prove Corollary 9.1.*

9.2 The Multiplicative Principle

The next principle is called the *multiplicative* principle and it can be illustrated as follows: imagine that you are given two postal stamps and three envelopes, how many ways to pack the letter? The answer is obviously $2 \cdot 3 = 6$.

Theorem 9.2 (The Multiplicative Principle). *Let X and Y be finite sets. Then $|X \times Y| = |X| \times |Y|$.*

Proof. If one of the sets X and Y are empty, then $X \times Y$ is empty as well and the statement follows.

Assume that none of the sets is empty. Let $|X| = n$, $|Y| = m$, and $f : [n] \rightarrow X$ and $g : [m] \rightarrow Y$ be bijections. Note that $\bigcup_{i=1}^n (\{f(i)\} \times Y) = X \times Y$. Additionally, note that $(\{f(i)\} \times Y) \cap (\{f(j)\} \times Y) = \emptyset$ for $i \neq j$. Finally, it is easy to see that $g_i : [m] \rightarrow (\{f(i)\} \times Y)$ such that $g_i(j) = (f(i), g(j))$ is a bijection. Hence, $|X \times Y| = \sum_{i=1}^n |\{f(i)\} \times Y| = n \cdot m$. □

Exercise 9.2. *Find the cardinality of the set $\{(x, y) : x, y \in [9] \text{ and } x \neq y\}$.*

By analogy with unions and intersections of many sets we can define the cross product of many sets. Let A_1, \dots, A_n be some sets. Then $\times_{i=1}^1 A_i = A_1$ and $\times_{i=1}^{k+1} A_i = \left(\times_{i=1}^k A_i\right) \times A_{k+1}$.

Corollary 9.2. *Let X_1, \dots, X_n be some finite sets. Then $|\times_{i=1}^n A_i| = \prod_{i=1}^n |X_i|$.*

Exercise 9.3. *Prove Corollary 9.2.*

Theorem 9.3. *For any set X , $|2^X| = 2^{|X|}$.*

Proof. By Corollary 8.1, $|2^X| = |\{0, 1\}^{|X|}|$, so it is enough to prove that $|\{0, 1\}^{|X|}| = 2^{|X|}$. This statement is true by Corollary 9.2 since $|\{0, 1\}^{|X|}| = \prod_{i=1}^{|X|} |\{0, 1\}| = 2^{|X|}$. □

¹Note that cross product is not associative and different definitions of the product of several sets are not equivalent. However, the bijection constructed in the previous section allow us to think about these definitions as if they are equivalent.

9.3 The Inclusion-exclusion Principle

The last principle we are going to discuss in this chapter is the inclusion-exclusion principle which helps us to find the size of the union of sets when they are not disjoint.

Theorem 9.4 (The Inclusion-exclusion Principle). *Let X and Y be finite sets. Then $|X \cup Y| = |X| + |Y| - |X \cap Y|$.*

Proof. Note that $X \cup Y = (X \setminus Y) \cup (Y \setminus X) \cup (X \cap Y)$. Hence, $|X \cup Y| = |X \setminus Y| + |Y \setminus X| + |X \cap Y|$. But it is possible to note that $|Y \setminus X| + |X \cap Y| = |Y|$ and $|X \setminus Y| + |X \cap Y| = |X|$. \square

Corollary 9.3. *Let X_1, \dots, X_n be some finite sets. Then*

$$\left| \bigcup_{i=1}^n X_i \right| = \sum_{S \subseteq [n] : S \neq \emptyset} (-1)^{|S|+1} \left| \bigcap_{i \in S} X_i \right|.$$

Proof. As always, we prove this statement using induction by n . The base case for $n = 2$ is true by Theorem 9.4.

By the induction hypothesis,

$$\left| \bigcup_{i=1}^k X_i \right| = \sum_{S \subseteq [k] : S \neq \emptyset} (-1)^{|S|+1} \left| \bigcap_{i \in S} X_i \right|.$$

In addition, by Theorem 9.4,

$$\left| \bigcup_{i=1}^{k+1} X_i \right| = \left| \bigcup_{i=1}^k X_i \right| + |X_{k+1}| - \left| \left(\bigcup_{i=1}^k X_i \right) \cap X_{k+1} \right|.$$

We need to simplify two elements of the sum on the right of the equality. By the induction hypothesis,

$$\left| \bigcup_{i=1}^k X_i \right| = \sum_{S \subseteq [k] : S \neq \emptyset} (-1)^{|S|+1} \left| \bigcap_{i \in S} X_i \right|.$$

In addition, it is easy to note that

$$\left| \left(\bigcup_{i=1}^k X_i \right) \cap X_{k+1} \right| = \left| \bigcup_{i=1}^k (X_i \cap X_{k+1}) \right|.$$

Thus using the induction hypothesis,

$$\begin{aligned} \left| \left(\bigcup_{i=1}^k X_i \right) \cap X_{k+1} \right| &= \\ &= \sum_{S \subseteq [k] : S \neq \emptyset} (-1)^{|S|+1} \left| \bigcap_{i \in S} (X_i \cap X_{k+1}) \right| = \\ &= \sum_{S \subseteq [k+1] : (k+1) \in S \text{ and } S \neq \{k+1\}} (-1)^{|S|} \left| \bigcap_{i \in S} X_i \right|. \end{aligned}$$

As a result,

$$|X_{k+1}| - \left| \left(\bigcup_{i=1}^k X_i \right) \cap X_{k+1} \right| = \sum_{S \subseteq [k+1] : (k+1) \in S} (-1)^{|S|+1} \left| \bigcap_{i \in S} X_i \right|.$$

Which implies that

$$\begin{aligned} \left| \bigcup_{i=1}^{k+1} X_i \right| &= \sum_{S \subseteq [k] : S \neq \emptyset} (-1)^{|S|+1} \left| \bigcap_{i \in S} X_i \right| + \\ &= \sum_{S \subseteq [k+1] : (k+1) \in S} (-1)^{|S|+1} \left| \bigcap_{i \in S} X_i \right| = \\ &= \sum_{S \subseteq [k+1] : S \neq \emptyset} (-1)^{|S|+1} \left| \bigcap_{i \in S} X_i \right|. \end{aligned}$$

□

End of The Chapter Exercises

- 9.4** How many numbers from [999] are not divisible neither by 3, nor by 5, nor by 7.
- 9.5** How many numbers x from 1 to 999 such that at least one of the digits of x is 7?
- 9.6** Let A, B be some finite sets such that $A \subseteq B$. Show that $|A \setminus B| = |A| - |B|$.
- 9.7** Let n be some positive integer. Find the cardinality of the set

$$\{(A, B) : A, B \subseteq [n] \text{ and } A \cap B \neq \emptyset\}?$$