Name: ____________________

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1. Show that $\sum_{i=1}^{n}(i+1)2^i = n2^{n+1}$ for all positive integers n.

Solution: We prove the statement using induction by n. The base case for $n = 1$ is true since $4 = (1+1)\cdot 2 = 1 \cdot 2^{1+1} = 4$.

Now we need to prove the induction step. Let us assume that $\sum_{i=1}^{n}(i+1)2^i = n2^{n+1}$. Note that $\sum_{i=1}^{n+1}(i+1)2^i = \sum_{i=1}^{n}(i+1)2^i + (n+2)2^{n+1} = n2^{n+1} + (n+2)2^{n+1} = (2n+2)2^{n+1} = (n+1)2^{n+2}$.

Hence, by the induction principle $\sum_{i=1}^{n}(i+1)2^i = n2^{n+1}$ for any positive integer n.

2. Let n be a positive integer and $A_1, \ldots, A_n$ be some sets. Let us define union of these sets as follows:

   1. $\cup_{i=1}^{1} A_i = A_1$,
   2. $\cup_{i=1}^{k+1} A_i = (\cup_{i=1}^{k} A_i) \cup A_{k+1}$.

   Show that $\cup_{i=1}^{n} [i] = [n]$.

   Solution: We prove the statement using induction by n. If $n = 1$, the statement is true since $\cup_{i=1} [i] = [1] = [1]$.

   To prove the induction step we assume that $\cup_{i=1}^{n} [i] = [n]$. By the definition of the union $\cup_{i=1}^{n+1} [i] = (\cup_{i=1}^{n} [i]) \cup [n+1]$. Hence, $\cup_{i=1}^{n+1} [i] = [n] \cup [n+1] = [n+1]$.

   Therefore by the induction principle, $\cup_{i=1}^{n} [i] = [n]$ for any positive integer n.

3. (10 points) Let Ω be some set. Consider $A_1, \ldots, A_n \subseteq \Omega$. Show that $\cup_{i=1}^n A_i = \{x \in \Omega \,:\, \exists i \in [n]\ x \in A_i\}$.

Solution: We prove the statement using induction by n. For $n = 1$ the statement is clearly true. Let us now prove the induction step from n to $n+1$. Assume that $\cup_{i=1}^n A_i = \{x \in \Omega \,:\, \exists i \in [n]\ x \in A_i\}$. Note that $\cup_{i=1}^{n+1} A_i = (\cup_{i=1}^n A_i) \cup A_{n+1} = \{x \in \Omega \,:\, \exists i \in [n]\ x \in A_i\} \cup A_{n+1}$. We denote $\{x \in \Omega \,:\, \exists i \in [n]\ x \in A_i\}$ by B. By the definition of union, $B \cup A$ is the set of all x such that either $x \in B$ or $x \in A_{n+1}$; therefore

$$B \cup A = \{x \in \Omega \,:\, (\exists i \in [n]\ x \in A_i) \text{ or } x \in A_{n+1}\} = \{x \in \Omega \,:\, \exists i \in [n+1]\ x \in A_i\}\,.$$

Which finishes the proof.