Name: ______________________

Pid: ______________________

1. (10 points) Let us consider a signature $(I, <; 0)$, where I is a unary relation intended to mean "is interesting", $<$ is a binary relation intended to mean "is less than", and 0 is a constant (a function with zero arguments).

   Translate into this language the English sentences listed below. If the English sentence is ambiguous, you will need more than one translation.

   - Zero is less than any number.
   - If any number is interesting, then zero is interesting.
   - No number is less than zero.
   - Any uninteresting number with the property that all smaller numbers are interesting certainly is interesting.
   - There is no number such that all numbers are less than it.
   - There is no number such that no number is less than it.

Solution: Translations of these phrases are the following:

- $\forall x \; 0 < x$,
- this time there are two possible interpretations of this phrase, $\forall x \; (I(x) \implies I(0))$ and $(\forall x \; I(x) \implies I(0))$,
- $\neg(\exists x \; x < 0)$,
- $\forall x \; (\neg I(x) \wedge \forall y \; (y < x \implies I(y)) \implies I(x)$,
- $\neg(\exists x \; \forall y \; y < x)$,
- $\neg(\exists x \; \neg(\exists y \; y < x))$.

2. (10 points) Let us consider a signature $\mathcal{S} = (=; +, \cdot)$, where predicates and functions are binary. Let $\mathfrak{M} = (\mathbb{N}; =; +, \cdot)$ be a structure.

   - Write a formula ϕ depending on x such that for any assignment s, $\mathfrak{M} \models \phi[s]$ iff $s(x) = 1$.
   - Write a formula ϕ depending on x and y such that for any assignment s, $\mathfrak{M} \models \phi[s]$ iff $s(x) \leq s(y)$.

Solution:

- To solve this exercise we need to note that the only integer x such that $xy = y$ for all $y \in \mathbb{N}$ is $x = 1$. Hence, we may consider the formula ϕ equal to $\forall y \; x \cdot y = y$.
- Note that all the numbers are positive; hence, for any $x, y \in \mathbb{N}$, there is z such that $x + z = y$ iff $x < y$. Therefore, we may consider ϕ equal to $\exists z \; x + z = y$.