Lecture 1 Mathematical Induction

Theorem (Induction Principle)

Let $P(n)$ be some statement about a positive integer n.

$P(n)$ is true for all positive integers n iff

- $P(1)$ is true and
- $P(k)$ implies $P(k+1)$.

base case

Example Show that $\int_0^{+\infty} x^n e^{-x}\, dx = n!$

Proof Let's start from proving the base case; i.e.

$$\int_0^{+\infty} x e^{-x}\, dx = 1$$

Note that $\int_0^{+\infty} x e^{-x} = x(-e^{-x})\Big|_0^{+\infty} - \int_0^{+\infty} 1 \cdot (-e^{-x}) =$

$$= 0 + \int_0^{+\infty} e^{-x}\, dx = (-e^{-x})\Big|_0^{+\infty} = 1$$

Now we are ready to prove the induction step from n to $n+1$.

The induction hypothesis is that

$$\int_0^{+\infty} x^n e^{-x}\, dx = n!$$

$$\int_0^{+\infty} x^{n+1} e^{-x} = x^{n+1}(-e^{-x})\Big|_0^{+\infty} - \int_0^{+\infty} (n+1)\, x^n e^{-x}\, dx =$$

$$= 0 + (n+1)\int_0^{+\infty} x^n e^{-x}\, dx \underset{\substack{\uparrow \\ \text{by the I.H.}}}{=} (n+1)!$$

Exercise Show that

$$\sum_{k=1}^{n} k\cdot k! = (n+1)!-1$$

The induction step is clear since

$1\cdot 1! = 2\cdot 1-1.$

Let us prove the induction step.

By the induction hypothesis, $\sum_{k=1}^{n} k\cdot k! = (n+1)!-1$

Hence, $\sum_{k=1}^{n+1} k\cdot k! = \sum_{k=1}^{n} k\cdot k! + (n+1)(n+1)! =$

$= (n+1)! - 1 + (n+1)(n+1)! = (n+1)!\,(n+1+1) - 1 =$

$= (n+2)! - 1$

Exercise Show that

$$\sum_{i=1}^{n} \frac{1}{i^2} \leq 2$$

It's enough to show that $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}$ since $2 - \frac{1}{n} < 2$.

We prove this using induction by n.

The base case is true since $\frac{1}{1} \leq 2 - 1$.

Let us now prove the induction step from n to $n+1$. By the induction hypotesis, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}$. Note that $\sum_{i=1}^{n+1} \frac{1}{i^2} =$

$$= \sum_{i=1}^{n} \frac{1}{i^2} + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n} + \frac{1}{(n+1)^2} =$$

$$= 2 - \left(\frac{n^2+2n+1 - n}{n(n+1)^2} \right) = 2 - \frac{1}{n+1} \left(\frac{n^2+n+1}{n(n+1)} \right) \leq$$

$$\leq 2 - \frac{1}{n+1}$$