Lecture 17

Theorem

Let Σ be a set of prop. formulas on $\Omega = \{x_1, \dots x_n \dots\}$, and φ be a prop. form on Ω.

If $\Sigma \models \varphi$, then there is a finite $\Sigma' \subseteq \Sigma$ s.t. $\Sigma' \models \varphi$.

Definition Let Σ be a set of formulas on Ω. Σ is satisfiable iff there is an assign. ρ to Ω s.t. $\psi|_\rho = T$ for any $\psi \in \Sigma$.

Theorem

maybe inf.

A set Σ of propos. formulas is satisfiable iff every finite subset is sat.

Proof

Let $\alpha_1, \ldots, \alpha_n \ldots$ be all prop. formulas on Ω.

we use here that $|\Omega| = \aleph_0$

A set Λ of prop. formulas on Ω is finitely sat. iff any finite $\Lambda' \subseteq \Lambda$ is satisfiable.

Define $\Delta_0 \dots \Delta_n \dots$ s.t.

$$\Delta_0 = \Sigma \qquad \Delta_{i+1} = \begin{cases} \Delta_i \cup \{\alpha_i\} & \text{if } \Delta_i \cup \{\alpha_i\} \text{ is fin. sat.} \\ \Delta_i \cup \{\neg\alpha_i\} & \text{otherwise.} \end{cases}$$

Assume $\Delta_i \cup \{\alpha_i\}$ isn't f.s.

it means that there is fin. $\Delta_i' \subseteq \Delta_i \cup \{\alpha_i\}$

s.t. Δ_i' is not sat.

Claim: $\Delta_i' \ni \alpha_i$

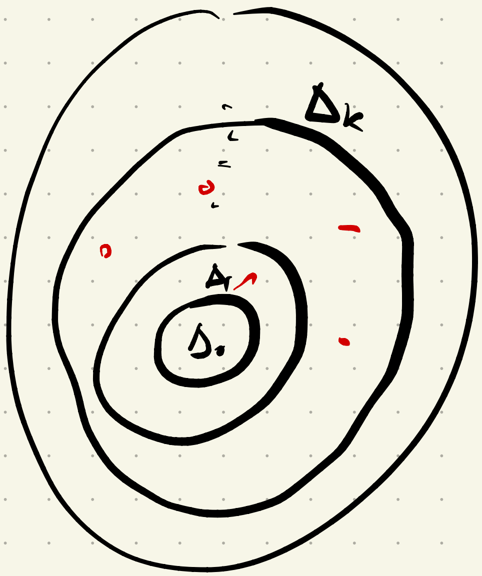
So $\Delta_i' = \Delta_i'' \cup \{\alpha_i\}$

I.e. if Δ_i'' is sat by ρ

α_i isn't.

Hence, $\Delta_i'' \cup \{\neg\alpha_i\}$ is sat.

$\Delta = \bigcup_{i \in \mathbb{N}} \Delta_i$ is finitely sat.



Notice that for any α_i
either $\alpha_i \in \Delta$
or $\neg\alpha_i \in \Delta$

Let us consider ρ s.t.

$\rho(x_i) = T$ iff $x_i \in \Delta$

Claim ρ satisfies Δ

Assume the opposite i.e., $\varphi/\rho = F$ $(\varphi \in \Delta)$

WLOG φ is a formula on $x_1 \ldots x_k$

Notice that $x_i^{\rho(x_i)} \in \Delta$ Hence,

$x_1^{\rho(x_1)} \ldots x_k^{\rho(x_k)}, \varphi$ is sat.

Theorem

Let Σ be a set of prop formulas on $\Omega = \{x_1 \ldots x_n \ldots\}$, and φ be a prop. form on Ω.

If $\Sigma \models \varphi$, then there is a finite $\Sigma' \subseteq \Sigma$ s.t $\Sigma' \models \varphi$.

Proof Assume it's not true; i.e. for any $\Sigma' \not\models \varphi$. In other words there is an as. ρ s.t. $\varphi|_\rho = F$ but $\psi|_\rho = T$ for all $\psi \in \Sigma'$

Hence, $\Sigma' \cup \{\neg\varphi\}$ is sat for any finite $\Sigma' \subseteq \Sigma$

So $\Sigma \cup \{\neg\varphi\}$ is sat. Hence, $\Sigma \not\models \varphi$ which is a contrad.