Name: ____________________

Pid: ____________________

1. (10 points) Let us formulate the pigeonhole principle using propositional formulas. $\Omega = \{x_{1,1}, \ldots, x_{n+1,1}, x_{1,2} \ldots, x_{n+1,n}\}$ (informally $x_{i,j}$ is true iff the ith pigeon is in the jth hole). Consider the following propositional formulas on the variables from Ω.

   - L_i ($i \in [n+1]$) is equal to $\bigvee_{j=1}^{n} x_{i,j}$. (Informally this formula says that the ith pigeon is in a hole.)
   - R_j ($j \in [n]$) is equal to $\bigvee_{i_1=1}^{n+1} \bigvee_{i_2=i_1+1}^{n+1} (x_{i_1,j} \wedge x_{i_2,j})$. (Informally this formula says that there are two pigeons in the jth hole.)

   Show that there is a natural deduction proof of $\left(\bigwedge_{i=1}^{n+1} L_i\right) \implies (\bigvee_{i=1}^{n} R_i)$.

Solution: Let $\phi = \left(\bigwedge_{i=1}^{n+1} L_i\right) \implies (\bigvee_{i=1}^{n} R_i)$. Note that if we prove that $\phi|_\rho$ is true for any assignement ρ, then by completeness theorem there is a natural deduction proof of ϕ.

Therefore it is enough to show that $\phi|_\rho$ is true for any assignement ρ. Let us fix some ρ.

- If $\left(\bigwedge_{i=1}^{n+1} L_i\right)\Big|_\rho$ is false, then $\phi|_\rho$ is true.
- If $(\bigvee_{i=1}^{n} R_i)\Big|_\rho$ is false, then $\phi|_\rho$ is true.
- Hence, to finish the proof we need to show that any other case is impossible. Assume that $(\bigvee_{i=1}^{n} R_i)\Big|_\rho$ and $\left(\bigwedge_{i=1}^{n+1} L_i\right)\Big|_\rho$ are true.

  Note that the fact that the first formula is true guarantees that $\sum_{i=1} \sum j = 1\rho(x_{i,j}) \geq n + 1$. However, the second formula says that $\sum_{i=1} \sum j = 1\rho(x_{i,j}) \leq n$, which leads us to a contradiction.

2. (10 points) Let $\phi = \bigvee_{i=1}^{m} \lambda_i$ be a clause; we say that the width of the clause is equal to m. Let $\phi = \bigwedge_{i=1}^{\ell} \chi_i$ be a formula in CNF; we say that the width of ϕ is equal to the maximal width of χ_i for $i \in [\ell]$.

   Let $p_n : \{T, F\}^n \to \{T, F\}$ such that $p_n(x_1, \ldots, x_n) = T$ iff the set $\{i : x_i = T\}$ has odd number of elements. Show that any CNF representation of p_n has width at least n.

Solution: Let us assume the opposite; i.e., that there is a formula $\phi = \bigwedge_{i=1}^{m} C_i$ CNF such that $\phi|_{x_1=v_1,\ldots,x_n=v_n} = p_n(v_1, \ldots, v_n)$ for all $v_1, \ldots, v_n \in \{T, F\}^n$ and width of each C_i is less than n

Without loss of generality we may assume that $C_1 = x_1 \vee \cdots \vee x_k$ for $k < n$. Let us consider a propositional assignement ρ to the variables $x_1, \ldots, x_n$ such that $\rho_u(x_i) = \begin{cases} F & \text{if } i \leq k \\ u & \text{if } i = k+1 \\ F & \text{otherwise.} \end{cases}$ It is clear that $\phi|_{\rho_T} = \phi|_{\rho_F} = F$. However, $p_n(\rho_T(x_1), \ldots, \rho_T(x_n)) \neq p_n(\rho_F(x_1), \ldots, \rho_F(x_n))$; therefore, $\phi|_{\rho_T} \neq \phi|_{\rho_F}$. As a result, the assumption is wrong.

3. (10 points) Write a natural deduction derivation of $(W \vee Y) \implies (X \vee Z)$ from hypotheses $W \implies X$ and $Y \implies Z$.

Solution:

1	$W \implies X$	
2	$Y \implies Z$	
3	$\mid\ W \vee Y$	
4	$\mid\ \mid\ W$	
5	$\mid\ \mid\ X$	$\Rightarrow$E, 1, 4
6	$\mid\ \mid\ X \vee Z$	$\vee$I, 5
7	$\mid\ \mid\ Y$	
8	$\mid\ \mid\ Z$	$\Rightarrow$E, 2, 7
9	$\mid\ \mid\ X \vee Z$	$\vee$I, 8
10	$\mid\ X \vee Z$	$\vee$E, 3, 4–6, 7–9
11	$(W \vee Y) \implies (X \vee Z)$	$\Rightarrow$I, 3–10

4. (10 points) We say that a clause C can be obtained from clauses A and B using the *resolution* rule if $C = A' \vee B'$, $A = x \vee A'$, and $B = \neg x \vee B'$, for some variable x.

   We say that a clause C can be derived from clauses $A_1, \ldots, A_m$ using resolutions if there is a sequence of clauses $D_1, \ldots, D_\ell = C$ such that each D_i

   - is either obtained from clauses D_j and D_k for $j, k < i$ using the *resolution* rule, or
   - is equal to A_j for some $j \in [m]$, or
   - is equal to $D_j \vee E$ for some $j < i$ and a clause E.

   Show that if $A_1, \ldots, A_m$ semantically imply C, then C can be derived from clauses $A_1, \ldots, A_m$ using resolutions.

Solution: First we prove that if clauses $A_1, \ldots, A_n$ semantically imply $\perp$, then there is a derivation of $\perp$ from $A_1, \ldots, A_n$ using the resolution rule.

We prove this using the induction on the number k of variables used in clauses $A_1, \ldots, A_n$. The base case for $k = 0$ is clear since in this case $A_i = \perp$ for all $i \in [n]$.

Let us now prove the induction step from k to $k+1$. We fix a variable x that is used by clauses $A_1, \ldots, A_n$. Let us split set of clauses $A_1, \ldots, A_n$ into three groups:

- the clauses $x \vee B_1, \ldots, x \vee B_p$ (i.e., the clauses that contain x),
- the clauses $\neg x \vee C_1, \ldots, \neg x \vee C_q$ (i.e., the clauses that contain $\neg x$),
- the clauses $D_1, \ldots, D_r$ (i.e., the clauses that neither contain x, nor $\neg x$).

Note that any assignment that sets x to be equal to T cannot make all A_i to be true since $A_1, \ldots, A_n$ semantically imply $\perp$. Therefore, $C_1, \ldots, C_q, D_1, \ldots, D_r$ semantically imply $\perp$. However, by the induction hypothesis, there is a derivation of $\perp$ from $C_1, \ldots, C_q, D_1, \ldots, D_r$ using the resolution rule. It is easy to see that this imply that there is a derivation of $\neg x$ from $\neg x \vee C_1, \ldots, \neg x \vee C_q$, $D_1, \ldots, D_r$. Similarly, there is a derivation of x from $x \vee B_1, \ldots, x \vee B_p, D_1, \ldots, D_r$. Hence, using the resolution rule we can derive $\perp$ from $A_1, \ldots, A_n$.