Given $a_0 = 100$ $a_{n+1} = 2a_n - 100.$

Find an explicit formula for a_n

spoiler: $a_n = 100$

Definition

Let $\{c_n\}_{n\geq 0}$ be a seq. of reals.

Then the (ordinary) generating function for $\{c_n\}_{n\geq 0}$ is $F(x) = \sum_{n\geq 0} c_n x^n$

$$a_{n+1} x^{n+1} = 2a_n x^{n+1} - 100x^{n+1}$$

$$\sum_{n\geq 0} a_{n+1} x^{n+1} = 2x \sum_{n\geq 0} a_n x^n - 100x \sum_{n\geq 0} x^n$$

$$\sum_{n\geq 0} a_{n+1} x^{n+1} - 2x \sum_{n\geq 0} a_n x^n - 100x \sum_{n\geq 0} x^n$$

Let $F(x)$ be the gen. function for $\{a_n\}_{n\geq 0}$

$$F(x) - a_0 = 2x\, F(x) - 100x \sum_{n\geq 0} x^n$$

$$F(x) - a_0 = 2x\, F(x) - \frac{100x}{1-x}$$

$$F(x)\,(1-2x) \quad - \quad 100x$$

$$F(x) = \frac{a_0}{1-2x} - \frac{100x}{(1-x)(1-2x)}$$

$$F(x) = 100 \sum_{n\geq 0} 2^n x^n - \frac{100x}{(1-x)(1-2x)}$$

We need to find A, B s.t.

$$\frac{A}{1-x}+\frac{B}{1-2x}=\frac{100x}{(1-x)(1-2x)}$$

$$A-2Ax+B-Bx=100x$$

$$\begin{cases} A+B=0 \\ -2 \end{cases}$$

$$\downarrow$$

$$\begin{cases} A=-B \\ 2B-B=100 \end{cases} \nearrow \begin{cases} A=-100 \\ B=100 \end{cases}$$

$$F(x)=10\sum_{n\geq 0}2^n x^n+\frac{100}{1-x}-\frac{100}{1-2x}=\sum\left(2^n\cdot 100+100-2^n\cdot 100\right)\cdot x^n$$

$$F(x)=\sum_{n\geq 0}100x^n.$$ Therefore $a_n=100$

Exercise Find an explicit formula

$a_{n+1} = 2a_n - 100 \quad a_0 = 200$

The answer is $2^n \cdot 100 + 100$

Let $a_{n+1} = 2a_n + n$ and $a_0 = 1$

$$a_{n+1} x^{n+1} = 2x\, a_n x^n + n x^{n+1}$$

$$\sum a_{n+1} x^{n+1} = 2x \sum a_n x^n + x \sum n x^n$$

Let $F(x)$ be the gen. function for a_n

$$F(x) - a_0 = 2x F(x) + x \sum_{n \geq 0} n x^n$$

$$F(x) - a_0 = 2x F(x) + x^2 \sum n x^{n-1}$$

$$F(x)(1-2x) = a_0 + x^2 \left(\frac{1}{1-x}\right)' = a_0 + x^2 \frac{1}{(1-x)^2}$$

$$F(x) = \frac{a_0}{1-2x} + \frac{x^2}{(1-x)^2\,(1-2x)}$$

$$F(x) = \frac{1-2x+x^2+x^2}{(1-2x)(1-x)^2} = \frac{1-2x+2x^2}{(1-2x)\,(1-x)^2}$$

$$\frac{A}{(1-x)^2} + \frac{B}{1-x} + \frac{C}{1-2x} = \frac{1-2x+2x^2}{(1-2x)\,(1-x)^2}$$

$$A + B - Bx + C - 2Cx + Cx^2 = 1 - 2x + 2x^2$$

$$\begin{cases} A + B + C = 1 \\ -B - 2C = -2 \\ C = 2 \end{cases} \longrightarrow \begin{cases} A = 1 \\ B = -2 \\ C = 2 \end{cases}$$